Higher Algebra I — Fall 2005

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Problem 1: Let R be a commutative ring with $1 \neq 0$. Let J be the nilradical of R. This is the the ideal consisting of the nilpotent elements of R, i.e., the set of elements $a \in R$ such that $a^n = 0$ for some nonnegative integer n. (Cf. Problem 1 on sheet 8.)

(a) Prove that J is contained in the intersection of all maximal ideals I of R:

$$J \subset \bigcap_{I \text{ maximal}} I.$$

(b) Prove that J equals the intersection of all prime ideals I of R:

$$J = \bigcap_{I \text{ prime}} I.$$

(Hint: Zorn's Lemma).

Problem 2: Let R be an integral domain and assume that every *prime* ideal in R is principal. Prove that every ideal of R is principal, i.e., R is a principal ideal domain. (Hint: Zorn's Lemma).

Problem 3: Let R be a unique factorization domain. Assume that for any pair of elements a, $b \in R$, the ideal $I = (a, b)$ they generate is a principal ideal. Prove that R is a principal ideal domain.

Problem 4: Let R be a principal ideal domain.

(a) Let $a_1, \ldots, a_n \in R$. Prove that for a greatest common divisor d of $a_1, \ldots, a_n$ there exist elements $x_1, \ldots, x_n \in R$ such that

$$d = x_1 a_1 + \cdots + x_n a_n.$$

(b) Let a, b, $c \in R$. Show that if a and b are relative prime that $b|ac$ implies $b|c$.