Higher Algebra I — Fall 2005

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Problem 1: (a) Let R be an integral domain, $a \in R^*$ and $b \in R$. Prove that a polynomial $f \in R[x]$ is irreducible in $R[x]$ if and only if the polynomial $f(ax+b)$ is irreducible in $R[x]$.

(b) Let p be a prime. Prove that the polynomial $f = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbf{Q}[x]$. (Hint: Consider $f(x+1)$ and use (a) and Eisenstein's criterion.)

Problem 2: Prove that $f = x^4 + 1$ is irreducible in $\mathbf{Q}[x]$. (Hint: The following procedure works for all polynomial f $\mathbf{Z}[x]$. If $g|f$ one can assume $\deg g \leq [\frac{\deg f}{2}] = s$ and one has $g(a)|f(a)$ for all $a \in \mathbf{Z}$. Choose different a_0, $\ldots$, $a_s \in \mathbf{Z}$ with $f(a_i) \neq 0$. For all $s+1$-tupels $(b_0, \ldots, b_s)$ of integers such that $b_i|f(a_i)$ choose the polynomial $g \in \mathbf{Z}[x]$ of degree $\leq s$ with $g(a_i) = b_i$ and check if $g|f$. Use symmetries to simplify the computations.)

Problem 3: Let m and n be natural numbers. Determine the structure of the following four $\mathbf{Z}$-modules:

(a) $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z})$,
(b) $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z})$,
(c) $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$,
(d) $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$.

Problem 4: List which of the 13 numbered Lemmas, Propositions, Thoerems and Corollaries in Section 3.1 remain true if one takes R-modules for arbitrary rings R with 1. Formulate modified versions if necessary.

Problem 5*: Find a counter-example to one of the numbered results of Section 3.1 for non commutative rings R.