# Higher Algebra I — Fall 2005

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**Problem sheet 2 September 1, 2005**

**Problem 1:** Prove the following remark from the lecture about the centralizer and normalizer of a subset $S$ of a group $G$:

(a) $\mathrm{Cent}(S) < \mathrm{Nor}(S) < G$.

(b) $H < G$ implies $H \triangleleft \mathrm{Nor}(H)$.

(c) $K < G$ and $H \triangleleft K$ implies $K < \mathrm{Nor}(H)$, i.e., $\mathrm{Nor}(H)$ is the largest subgroup of $G$ in which $H$ is normal.

**Problem 2:** Let $c : G \longrightarrow \mathrm{Perm}(G)$ be the map defined for $a \in G$ by $c(a) : G \to G$, $g \mapsto aga^{-1}$.

(a) Show that $c$ is a homomorphism from $G$ into $\mathrm{Aut}(G) < \mathrm{Perm}(G)$, the subgroup of automorphisms of $G$.

(b) Is the image of $c$ normal in $\mathrm{Aut}(G)$?

(c) Determine the kernel of $c$.

**Problem 3:** (a) Let $G$ be group and $H$ be a group of finite index. Show that there exists a normal subgroup $N$ of $G$ contained in $H$ and also of finite index. (Hint: If $[G : H] = n$, find a homomorphism of G into $S_n$ (the group of permutations of the set $\{1, \ldots, n\}$) whose kernel is contained in $H$.)

(b) Let G be a group and let $H_1$, $H_2$ be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

**Problem 4:** Determine a composition series of the group $S_4$, the group of permutations of the set $\{1, 2, 3, 4\}$.