

Higher Algebra I — Fall 2005

Gerald Hoehn

Problem sheet 3 September 8, 2005

Problem 1: Let G_1, \dots, G_n be finite simple groups. Show that G_1, \dots, G_n are the composition factors of the direct product $G_1 \times \dots \times G_n$.

Problem 2: Show that if the automorphism group of the finite cyclic group of order n is itself cyclic that n must be contained in the set $S = \{2, 4, p^k, 2p^k\}$ with p an odd prime and k a positive integer. (Hint: Show $\mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/p_1^{a_1}\mathbf{Z} \times \dots \times \mathbf{Z}/p_l^{a_l}\mathbf{Z}$ where $p_1^{a_1} \dots p_l^{a_l}$ is the prime factorization of n and use this to investigate the $\varphi(n) = \varphi(p_1^{a_1}) \dots \varphi(p_l^{a_l}) = (p_1^{a_1-1}(p_1-1)) \dots (p_l^{a_l-1}(p_l-1))$ generators of $\mathbf{Z}/n\mathbf{Z}$. Identify the automorphism group of $\mathbf{Z}/n\mathbf{Z}$ with the *multiplicative* group $(\mathbf{Z}/n\mathbf{Z})^\times$ of residue classes modulo n of the $\varphi(n)$ non-negative integers $m < n$ which are relatively prime to n .)

Problem 3: If K is a normal subgroup of the group G and K is cyclic, prove that the commutator G' is a subgroup of the centralizer $C(K)$ of K . (Hint: Use from Problem 2 that the automorphism group of a cyclic group is abelian.)

Problem 4: Let G be a finite solvable group. Prove that there exists a chain $G = N_0 > N_1 > \dots > N_n = \{e\}$ of subgroups of G such that N_i is normal in G and N_i/N_{i+1} is abelian for $i = 0, 1, \dots, n-1$. (Hint: Prove that a minimal nontrivial normal subgroup M of G is necessarily abelian and use induction. To see that M is abelian, let N be a normal subgroup of M of prime index and show that $[x, y] \in N$ for all $x, y \in M$. Apply the same argument to gNg^{-1} , $g \in G$, to show that $[x, y]$ lies in the intersection of all G -conjugates of N , and use minimality of M to conclude that $[x, y] = e$.)

Problem 5*: Show that the automorphism group of the finite cyclic group of order n is itself cyclic if n is contained in the set $S = \{2, 4, p^k, 2p^k\}$ with p an odd prime and k a positive integer.