Higher Algebra I — Fall 2005

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Problem 1: Let $G_1, \ldots, G_n$ be finite simple groups. Show that $G_1, \ldots, G_n$ are the composition factors of the direct product $G_1 \times \cdots \times G_n$.

Problem 2: Show that if the automorphism group of the finite cyclic group of order n is itself cyclic that n must be contained in the set $S = \{2,\, 4,\, p^k,\, 2\,p^k\}$ with p an odd prime and k a positive integer. (Hint: Show $\mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/p_1^{a_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p_l^{a_l}\mathbf{Z}$ where $p_1^{a_1} \cdots p_l^{a_l}$ is the prime factorization of n and use this to investigate the $\varphi(n) = \varphi(p^{a_1}) \cdots \varphi(p^{a_l}) = (p_1^{a_1-1}(p_1-1)) \cdots (p_l^{a_1-1}(p_l-1))$ generators of $\mathbf{Z}/n\mathbf{Z}$. Identify the automorphism group of $\mathbf{Z}/n\mathbf{Z}$ with the *multiplicative* group $(\mathbf{Z}/n\mathbf{Z})^\times$ of residue classes modulo n of the $\varphi(n)$ non-negative integers $m < n$ which are relatively prime to n.)

Problem 3: If K is a normal subgroup of the group G and K is cyclic, prove that the commutator G' is a subgroup of the centralizer $C(K)$ of K. (Hint: Use from Problem 2 that the automorphism group of a cyclic group is abelian.)

Problem 4: Let G be a finite solvable group. Prove that there exists a chain $G = N_0 > N_1 > \cdots > N_n = \{e\}$ of subgroups of G such that N_i is normal in G and N_i/N_{i+1} is abelian for $i = 0,\, 1,\, \ldots,\, n-1$. (Hint: Prove that a minimal nontrivial normal subgroup M of G is necessarily abelian and use induction. To see that M is abelian, let N be a normal subgroup of M of prime index and show that $[x, y] \in N$ for all $x,\, y \in M$. Apply the same argument to gNg^{-1}, $g \in G$, to show that $[x, y]$ lies in the intersection of all G-conjugates of N, and use minimality of M to conclude that $[x, y] = e$.)

Problem 5^***:** Show that the automorphism group of the finite cyclic group of order n is itself cyclic if n is contained in the set $S = \{2,\, 4,\, p^k,\, 2\,p^k\}$ with p an odd prime and k a positive integer.