Higher Algebra I — Fall 2005

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Problem 1: Let G be a finite group acting on a finite set X. For $g \in G$, let X^g be the fixpoints of g, i.e., $X^g = \{x \in X \mid gx = x\}$. Proof *Burnside's formula*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

for the number of G-orbits. (Hint: Count the subset $F = \{(g, x) \mid gx = x\} \subset G \times X$ in two different ways.)

Problem 2: Let N be a normal subgroup of a finite group G. Let $S \subset G$ be a conjugacy class of elements in G, and assume that $S \subset N$. Prove that S is a union of n conjugacy classes in N, all having the same cardinality, where n equals the index $[G : N.\mathrm{Cent}(x)]$ of the group generated by N and the centralizer $\mathrm{Cent}(x)$ in G of any element $x \in S$.

Problem 3: Let $\sigma \in S_n$ such that in the cycle decompostion of σ there are λ_i cycles of length i for $i = 1, \ldots, n$.

(a) Describe the centralizer of σ in S_n and determine its order.

(b) Describe the normalizer of the cyclic group $\langle \sigma \rangle$ generated by σ in S_n and determine its order.

(c) Determine the number of pairs of commuting elements in S_n, i.e., the number of pairs (σ, τ) with σ, $\tau \in S_n$ and $\sigma\tau = \tau\sigma$.

Problem 4: Proof that A_n, the alternating group of degree n, is simple for $n \geq 5$. (Hint: Use that A_n is generated for $n \geq 5$ by 3-cycles. Let $\sigma \in N \setminus \{e\}$ be an element of a normal subgroup N of A_n with a maximal number of fixpoints. Show that σ is conjugated to a 3-cycle or an element with more fixpoints. Distinguish the two cases that all orbits of $\langle \sigma \rangle$ have size 1 and 2 or that there is at least one orbit of size 3 or larger.)

Problem 5^* (the conjugacy classes of A_n): Let $T \subset S_n$ be a conjugacy class in S_n, i.e., the set of all permutations $\sigma \in S_n$ which have the same cycle type and assume that $T \subset A_n$. Show that each such conjugacy class T is either also a conjugacy class in A_n or decomposes in two conjugacy class of A_n of the same size; the second case happens if and only if the cycle type of an element of T consists of distinct odd integers.