

Higher Algebra I — Fall 2005

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Problem sheet 6 September 29, 2005

Problem 1 (ABELIAN GROUPS): (2 points)

- (a) Determine the number of non-isomorphic abelian groups of order 5715360000.
(b) Determine the number of subgroups of order 1000 of $\mathbf{Z}/1000\mathbf{Z} \times \mathbf{Z}/1000\mathbf{Z}$.

Problem 2 (PRESENTATIONS): (6 points)

Prove the following group presentations:

- (a) $\langle a, b \mid a^2b^{-2}, a^{-1}bab \rangle = Q_8$, the quaternion group of order 8 defined on sheet 1.
(b) $\langle \tau_1, \dots, \tau_{n-1} \mid \tau_i^2 \text{ for } 1 \leq i \leq n-1; (\tau_i\tau_{i+1})^3 \text{ for } 1 \leq i \leq n-2; (\tau_i\tau_j)^2 \text{ for } 1 \leq i, j \leq n-1 \text{ with } |i-j| \geq 2 \rangle = S_n$, the symmetric group of degree n . (Hint: τ_i is the transposition $(i \ i+1)$. Use induction on n .)
(c) $\langle a, b, c \mid bab^{-1}a^{-2}, cbc^{-1}b^{-2}, aca^{-1}c^{-2} \rangle = \{e\}$, the trivial group with one element.

Problem 3 (COPRODUCTS FOR ABELIAN GROUPS): (4 points)

Prove the existence of coproducts for abelian groups, i.e., prove Theorem 39 with each occurrence of the word “group” replaced by “abelian group”.

Problem 4 (PUSHOUTS): (4 points)

Let F, H and G be groups and $f : G \rightarrow F, h : G \rightarrow H$ be homomorphisms. Prove the existence of a group $F \amalg_G H$ (called the pushout of f and h) and homomorphisms $q_1 : F \rightarrow F \amalg_G H, q_2 : H \rightarrow F \amalg_G H$ with $q_1 \circ f = q_2 \circ h$ such that the following universal property is satisfied: For any group J and homomorphisms $j_1 : F \rightarrow J, j_2 : H \rightarrow J$ with $j_1 \circ f = j_2 \circ h$ there exists a homomorphism $\varphi : F \amalg_G H \rightarrow J$ such that $j_1 = q_1 \circ \varphi$ and $j_2 = q_2 \circ \varphi$.

Problem 5*: Show that the group $\langle S, T \mid S^2, (ST)^3 \rangle$ is isomorphic to the group $\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z}) / \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$, where $\mathrm{SL}_2(\mathbf{Z})$ is the group of 2×2 -matrices with integer entries and determinant 1. (Hint: Consider the action of $\mathrm{SL}_2(\mathbf{Z})$ by Möbius transformations on the upper halfplane $\mathbf{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$ of complex numbers given by $(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \frac{az+b}{cz+d}$; cf. the book by J.-P. Serre: *A course in Arithmetic*.)