Higher Algebra I — Fall 2005

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Problem sheet 6 September 29, 2005

Problem 1 (Abelian Groups): *(2 points)*

(a) Determine the number of non-isomorphic abelian groups of order 5715360000.

(b) Determine the number of subgroups of order 1000 of $\mathbf{Z}/1000\mathbf{Z} \times \mathbf{Z}/1000\mathbf{Z}$.

Problem 2 (Presentations): *(6 points)*

Prove the following group presentations:

(a) $\langle a,\, b \mid a^2b^{-2},\, a^{-1}bab\rangle = Q_8$, the quaternion group of order 8 defined on sheet 1.

(b) $\langle \tau_1,\, \ldots,\, \tau_{n-1} \mid \tau_i^2$ for $1 \leq i \leq n-1$; $(\tau_i\tau_{i+1})^3$ for $1 \leq i \leq n-2$; $(\tau_i\tau_j)^2$ for $1 \leq i,\, j \leq n-1$ with $|i-j| \geq 2\ \rangle = S_n$, the symmetric group of degree n. (Hint: τ_i is the transposition $(1\ i{+}1)$. Use induction on n.)

(c) $\langle a,\, b,\, c \mid bab^{-1}a^{-2},\, cbc^{-1}b^{-2},\, aca^{-1}c^{-2}\rangle = \{e\}$, the trivial group with one element.

Problem 3 (Coproducts for Abelian Groups): *(4 points)*

Prove the existence of coproducts for abelian groups, i.e., prove Theorem 39 with each occurrence of the word "group" replaced by "abelian group".

Problem 4 (Pushouts): *(4 points)*

Let F, H and G be groups and $f : G \longrightarrow F$, $h : G \longrightarrow H$ be homomorphisms. Prove the existence of a group $F \coprod_G H$ (called the pushout of f and h) and homomorphisms $q_1 : F \longrightarrow F \coprod_G H$, $q_2 : H \longrightarrow F \coprod_G H$ with $q_1 \circ f = q_2 \circ g$ such that the following universal property is satisfied: For any group J and homomorphisms $j_1 : F \longrightarrow J$, $j_2 : H \longrightarrow J$ with $j_1 \circ f = j_2 \circ g$ there exists a homomorphism $\varphi : F \coprod_G H \longrightarrow J$ such that $j_1 = q_1 \circ \varphi$ and $j_2 = q_2 \circ \varphi$.

Problem 5*: Show that the group $\langle S,\, T \mid S^2,\, (ST)^3\rangle$ is isomorphic to the group $\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\{\pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\}$, where $\mathrm{SL}_2(\mathbf{Z})$ is the group of 2×2-matrices with integer entries and determinant 1. (Hint: Consider the action of $\mathrm{SL}_2(\mathbf{Z})$ by Möbius transformations on the upper halfplane $\mathbf{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$ of complex numbers given by $(\left(\begin{smallmatrix}a & b\\ b & c\end{smallmatrix}\right), z) \mapsto \frac{az+b}{cz+d}$; cf. the book by J.-P. Serre: *A course in Arithmetic.*)