Higher Algebra I — Fall 2005

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Problem 1: Show that there are at most two non-isomorphic groups G of order pq, where p and q are primes with $p < q$. (Hint: G is either a direct or a semidirect product of its Sylow subgroups.)

Problem 2: Let G and F be groups and $\varphi : G \longrightarrow F$ be a surjective group homomorphism. Let $N = \ker \varphi$ be the kernel of φ. Prove that G is a semidirect product with normal subgroup N if and only if there is a homomorphism $s : F \longrightarrow G$ sich that $\varphi \circ s = \mathrm{id}_F$, the identity map on F. One says in this case that the short exact sequence

$$\{e\} \longrightarrow N \longrightarrow G \xrightarrow{\varphi} F \longrightarrow \{e\}$$

splits and s is called a *split* of φ.

Problem 3: Decide which of the following sets are subrings of the ring of all functions from the closed interval $[0, 1]$ to $\mathbf{C}$:

(a) The functions $f(x)$ such that $f(q) = 0$ for all $q \in [0, 1] \cap \mathbf{Q}$.

(b) The polynomial functions.

(c) The functions which have a finite number of zeros together with the zero function.

(d) The functions wich have infinitely many of zeros.

(e) All functions f such that $\lim_{x\to 1^-} f(x) = 0$.

(f) All rational linear combinations of the functions $\sin nx$ and $\cos nx$, where $n \in \mathbf{Z}$, $n \geq 0$.

(g) All functions which are equal to a convergent power series expanded in 0 with radius of convergence larger than 1.

(h) The set of continous nowhere differentiable functions together with the differentiable functions.

Problem 4: (a) Show that the four dimensional real vector space $\mathbf{R}^4$ with base e, i, j and k becomes a noncommutative ring if we use the vector space addition as ring addition and if we indroduce a $\mathbf{R}$-linear ring multiplication by requiring that e is a 1 and

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

(b) Show that in the above ring every element besides 0 has a multiplicative inverse.

Problem 5***:** Let n be a natural number n and p be a prime. Write

$$n = a_0 + a_1\, p + a_2\, p^2 + \cdots + a_k\, p^k$$

with integers a_i, $0 \leq a_i < p$ (this is the representation of n using the base p). Consider the regular action of $Z_p = \mathbf{Z}/p\mathbf{Z}$ on itself by left multiplication. Using this action, define $Z_p^{\wr r}$, $r \geq 1$, inductively by $Z_p^{\wr 1} = Z_p$, $Z_p^{\wr k+1} = Z_p^{\wr k} \wr Z_p$. Prove that any p-Sylow subgroup of S_n, the symmetric group of degree n, is isomorphic to

$$\underbrace{Z_p^{\wr 1} \times \cdots \times Z_p^{\wr 1}}_{a_1-\text{times}} \times \underbrace{Z_p^{\wr 2} \times \cdots \times Z_p^{\wr 2}}_{a_2-\text{times}} \times \cdots \times \underbrace{Z_p^{\wr k} \times \cdots \times Z_p^{\wr k}}_{a_k-\text{times}}.$$

(Hint: The exponent of p in the prime factor decomposition of $n!$ is

$$[\frac{n}{p}] + [\frac{n}{p^2}] + [\frac{n}{p^3}] + \cdots,$$

where $[x]$ denotes the largest integer $\leq x$.)