Higher Algebra I — Fall 2005

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Problem 1: A nilpotent element of a ring is an element a such that $a^n = 0$ for some nonnegative integer n.

(a) Show that the set of nilpotent elements of a commutative ring form an ideal — called the *nilradical*.

(b) Let R be a commutative ring with $1 \neq 0$. Prove that if $a \in R$ is nilpotent, $1 - ab$ is a unit for all $b \in R$.

Problem 2: Let I, J and K be ideals of a ring R. Prove:

(a) $I + J$ is the smallest ideal of R containing both I and J.

(b) $IJ \subset I \cap J$.

(c) $I(J + K) = IJ + IK$ and $(J + K)I = JI + KI$.

(d) If $J \subset I$ then $I \cap (J + K) = J + (I \cap K)$.

Problem 3: For an ideal I of a ring R denote with $M_n(I)$ the set of $n \times n$-matrices with entries in I. Prove that $I \mapsto M_n(I)$ defines a bijective map of the set of ideals of R onto the set of ideals of the ring $M_n(R)$.

Problem 4: Show that every ring R can be embedded into a ring S with 1. (Hint: Take $S = \mathbf{Z} \times R$ and define for m, $n \in \mathbf{Z}$ and a, $b \in R$ the addition by $(m, a) + (n, b) = (m + n, a + b)$ and the multiplication by $(m, a)(n, b) = (mn, mb + na + ab)$, where for a nonegative integer n one writes $na = \underbrace{a + \cdots + a}_{n-\text{times}}$.)