Higher Algebra I — Fall 2005

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Problem 1: Let R be a commutative ring with 1. The *ring of formal power series* $R[[x]]$ in the indeterminate x with coefficients from R, consists of formal infinite sums

$$\sum_{n=0}^{\infty} a_i\, x^i = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots.$$

The ring structure is given by extending the ring structure of the polynomial ring $R[x]$ to "polynomials of infinite degree" in the obvious way.

(a) Prove that $\sum_{n=0}^{\infty} a_i\, x^i$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.

(b) Prove that with R also $R[[x]]$ is an integral domain.

Problem 2: Let R be a commutative ring with 1. Prove that a polynomial $f \in R[x]$ is a zero-divisor in $R[x]$ if and only if there exists a $b \in R$ such that $bf = 0$.

Problem 3: Let D be a squarefree integer (i.e. an integer which has no integer square besides 1 as a divisor). Let

$$\omega = \begin{cases} \sqrt{D}, & \text{if } D \equiv 2,\, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

The set $\mathcal{O} = \{a + b\,\omega \mid a,\, b \in \mathbf{Z}\}$ is a subring of the real numbers (if D is positive) or complex numbers (if D is negative). Define the field norm of $x + y\sqrt{D}$, where $x,\, y \in \mathbf{Q}$ (or $\mathbf{R}$), by $N(x + y\sqrt{D}) = (x + y\sqrt{D})(x - y\sqrt{D}) = x^2 - Dy^2$.

Prove that for $D = -2,\, -3,\, -7$ or -11 the ring $\mathcal{O}$ is an Euclidean Domain with respect to N. (Hint: Modify the proof for $\mathbf{Z}[i]$ ($D = -1$) given in the lecture. For $D = -3,\, -7,\, -11$, prove that every element of $\mathbf{C}$ differs from an element of $\mathcal{O}$ by an element whose norm is at most $(1 + |D|)^2/(16|D|)$, which is less than 1 for these values of D. Plotting the points of $\mathcal{O}$ in $\mathbf{C}$ may be helpful.)

Problem 4: Let K be a field, let $a_1,\, \ldots,\, a_n$ be pairwise different and let $b_1,\, \ldots,\, b_n$ be arbitrary elements of K. Show that there is a unique polynomial $f \in K[x]$ with $\deg(f) \leq n - 1$ and $f(a_i) = b_i$ for $i = 1,\, \ldots,\, n$.