

# Higher Algebra 2 — Spring 2006

Gerald Hoehn

## Problem sheet 1      January 20, 2006

**Problem 1:** (a) Prove that any two bases in a vector space have the same cardinality. (*Hint:* Generalize the case of a vector space with finite a base (Theorem 2). Use the fact that a finite number of elements in one basis is contained in the space generated by a finite number of elements in another basis.)

(b) Prove that if the dimension of a vector space  $V$  is infinite, the cardinality of a basis in the dual space  $V^*$  is larger the the cardinality of a basis of  $V$ .

**Problem 2:** Let  $V$  be a vector space over a field  $K$ , and let  $U, W$  be subspaces. Show that

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

Here,  $U + W$  denotes the subspace of linear combinations of elements of  $U \cup W$ .

**Problem 3\*\*\*:** For a vector space  $V$  over a field  $K$ , denote by  $i_V : V \rightarrow V^{**}$  the canonical injection given by evaluation and by  $i_V^* : V^{***} \rightarrow V^*$  its adjoint.

(a) Prove that  $i_V^* \circ i_V : V^* \rightarrow V^*$  equals the identity map of  $V^*$ .

(b) For a base  $\mathcal{B}$  of  $V$  denote by  $\mathcal{B}^*$  be the dual base. Let  $f_{\mathcal{B}} : V \rightarrow V^*$  be the injection defined by using the base  $\mathcal{B}$  and  $f_{\mathcal{B}^*} : \text{im} f_{\mathcal{B}} \rightarrow (\text{im} f_{\mathcal{B}})^*$ . Prove that

$$\text{res}_{\text{im} f_{\mathcal{B}}}^{V^*} \circ i_V = f_{\mathcal{B}^*} \circ f_{\mathcal{B}},$$

where  $\text{res}_{\text{im} f_{\mathcal{B}}}^{V^*}$  is the natural restriction of a linear functional on  $V^*$  to  $\text{im} f_{\mathcal{B}}$ .

**Problem 4:** Determine the number of  $k$ -dimensional subspaces of an  $n$ -dimensional vector space over a field with  $q$  elements.