Higher Algebra 2 — Spring 2006

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Problem 1: (a) Prove that any two bases in a vector space have the same cardinality. (*Hint:* Generalize the case of a vector space with finite a base (Theorem 2). Use the fact that a finite number of elements in one basis is contained in the space generated by a finite number of elements in another basis.)

(b) Prove that if the dimension of a vector space V is infinite, the cardinality of a basis in the dual space V^* is larger the the cardinality of a basis of V.

Problem 2: Let V be a vector space over a field K, and let U, W be subspaces. Show that

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

Here, $U + W$ denotes the subspace of linear combinations of elements of $U \cup W$.

Problem 3*:** For a vector space V over a field K, denote by $i_V : V \longrightarrow V^{**}$ the canonical injection given by evaluation and by $i_V^* : V^{***} \longrightarrow V^*$ its adjoint.

(a) Prove that $i_V^* \circ i_{V^*} : V^* \longrightarrow V^*$ equals the identity map of V^*.

(b) For a base $\mathcal{B}$ of V denote by $\mathcal{B}^*$ be the dual base. Let $f_{\mathcal{B}} : V \longrightarrow V^*$ be the injection defined by using the base $\mathcal{B}$ and $f_{\mathcal{B}^*} : \mathrm{im} f_{\mathcal{B}} \longrightarrow (\mathrm{im} f_{\mathcal{B}})^*$. Prove that

$$\mathrm{res}^{V^*}_{\mathrm{im} f_{\mathcal{B}}} \circ i_V = f_{\mathcal{B}^*} \circ f_{\mathcal{B}},$$

where $\mathrm{res}^{V^*}_{\mathrm{im} f_{\mathcal{B}}}$ is the natural restriction of a linear functional on V^* to $\mathrm{im} f_{\mathcal{B}}$.

Problem 4: Determine the number of k-dimensional subspaces of an n-dimensional vector space over a field with q elements.